Hw 5 Solution

Problem 3.1.1 Solution

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -1 \\ (x+1)/2 & -1 \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1}$$

Each question can be answered by expressing the requested probability in terms of $F_X(x)$.

(a)

$$P[X > 1/2] = 1 - P[X < 1/2] = 1 - F_X(1/2) = 1 - 3/4 = 1/4 \tag{2}$$

(b) This is a little trickier than it should be. Being careful, we can write

$$P[-1/2 \le X < 3/4] = P[-1/2 < X \le 3/4] + P[X = -1/2] - P[X = 3/4] \tag{3}$$

Since the CDF of X is a continuous function, the probability that X takes on any specific value is zero. This implies $P[X = 3/4] = 0$ and $P[X = -1/2] = 0$. (If this is not clear at this point, it will become clear in Section 3.6.) Thus,

$$P[\ \ 1/2 \le X < 3/4] = P[\ \ 1/2 < X \le 3/4] = F_X(3/4) \quad F_X(\ \ 1/2) = 5/8 \tag{4}$$

(c)

$$P[|X| \le 1/2] = P[-1/2 \le X \le 1/2] = P[X \le 1/2] - P[X < -1/2] \tag{5}$$

Note that $P[X \le 1/2] = F_X(1/2) = 3/4$. Since the probability that $P[X = \ \ 1/2] = 0$, $P[X < -1/2] = P[X \le 1/2]$. Hence $P[X < -1/2] = F_X(-1/2) = 1/4$. This implies

$$P[|X| \le 1/2] = P[X \le 1/2] - P[X < -1/2] = 3/4 - 1/4 = 1/2 \tag{6}$$

(d) Since $F_X(1) = 1$, we must have $a \le 1$. For $a \le 1$, we need to satisfy

$$P[X \le a] = F_X(a) = \frac{a+1}{2} = 0.8 \tag{7}$$

Thus $a = 0.6$.

Problem 3.2.1 Solution

$$f_X(x) = \begin{cases} cx & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) From the above PDF we can determine the value of c by integrating the PDF and setting it equal to 1.

$$\int_0^2 cx\,dx = 2c = 1 \tag{2}$$

Therefore $c = 1/2$.

(b) $P[0 \leq X \leq 1] = \int_0^1 \frac{x}{2}\,dx = 1/4$

(c) $P[-1/2 \leq X \leq 1/2] = \int_0^{1/2} \frac{x}{2}\,dx = 1/16$

(d) The CDF of X is found by integrating the PDF from 0 to x.

$$F_X(x) = \int_0^x f_X(x')\,dx' = \begin{cases} 0 & x < 0 \\ x^2/4 & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases} \tag{3}$$

Problem 3.3.4 Solution

We can find the expected value of X by direct integration of the given PDF.

$$f_Y(y) = \begin{cases} y/2 & 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The expectation is

$$E[Y] = \int_0^2 \frac{y^2}{2}\,dy = 4/3 \tag{2}$$

To find the variance, we first find the second moment

$$E[Y^2] = \int_0^2 \frac{y^3}{2}\,dy = 2. \tag{3}$$

The variance is then $\text{Var}[Y] = E[Y^2] - E[Y]^2 = 2 - (4/3)^2 = 2/9$.

Problem 3.4.2 Solution

From Appendix A, we observe that an exponential PDF Y with parameter $\lambda > 0$ has PDF

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In addition, the mean and variance of Y are

$$E[Y] = \frac{1}{\lambda} \qquad \text{Var}[Y] = \frac{1}{\lambda^2} \tag{2}$$

(a) Since $\text{Var}[Y] = 25$, we must have $\lambda = 1/5$.

(b) The expected value of Y is $E[Y] = 1/\lambda = 5$.

(c)

$$P[Y > 5] = \int_5^\infty f_Y(y)\, dy = -e^{-y/5}\Big|_5^\infty = e^{-1} \tag{3}$$

Problem 3.5.1 Solution

Given that the peak temperature, T, is a Gaussian random variable with mean 85 and standard deviation 10 we can use the fact that $F_T(t) = \Phi((t-\mu_T)/\sigma_T)$ and Table 3.1 on page 123 to evaluate the following

$$P[T > 100] = 1 - P[T \le 100] = 1 - F_T(100) = 1 - \Phi\left(\frac{100-85}{10}\right) \tag{1}$$

$$= 1 - \Phi(1.5) = 1 - 0.933 = 0.066 \tag{2}$$

$$P[T < 60] = \Phi\left(\frac{60-85}{10}\right) = \Phi(-2.5) \tag{3}$$

$$= 1 - \Phi(2.5) = 1 - .993 = 0.007 \tag{4}$$

$$P[70 \le T \le 100] = F_T(100) - F_T(70) \tag{5}$$

$$= \Phi(1.5) - \Phi(-1.5) = 2\Phi(1.5) - 1 = .866 \tag{6}$$